CH-4

Transient circuit Analysis

Terminology

First Order circuit:- Only contain 1 energy storage elements
- their Vtg or currents are described by 1st ODE.

- Two types of 1st order CKts
  ① RL circuit
  ② RC circuit

Natural Response:- How the CKts respond when a switch is opened or closed.

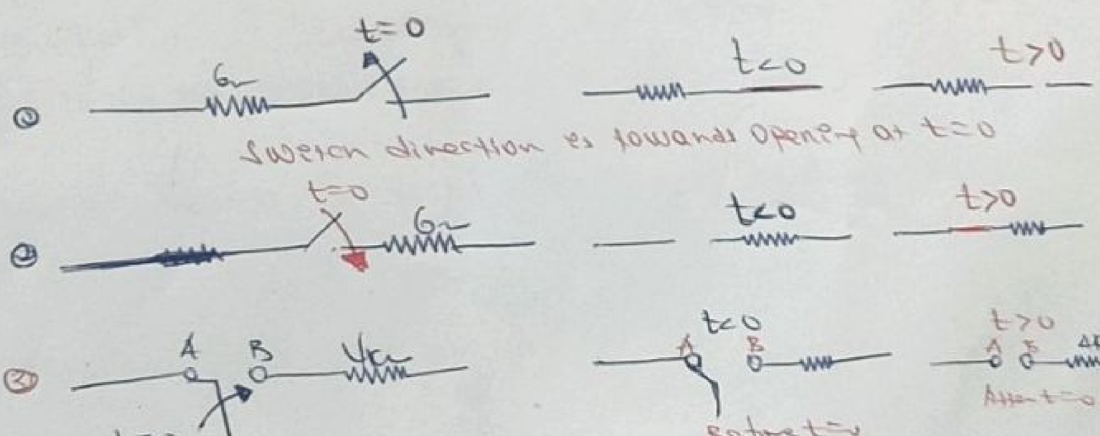
- RC and RL CKts:- the equation to analyze RC or RL CKt is 1st ODE. Thus, RC and RL CKt are known as 1st-order CKt.

  - To analyze the resistor (resistive) CKt, algebraic eqns are needed.
  - To analyze C and L time-differential equations are involved.

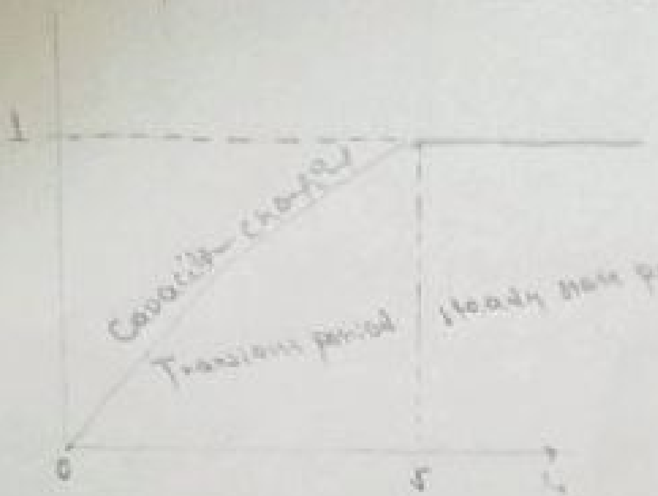
Sources:-

- **Source-free CKts:-** initial conditions
- The energy is initially stored in C or L, and the energy causes current to flow in the CKt and is gradually dissipated in resistors.
- **Independent sources:-** DC, sinusoidal & exponential source

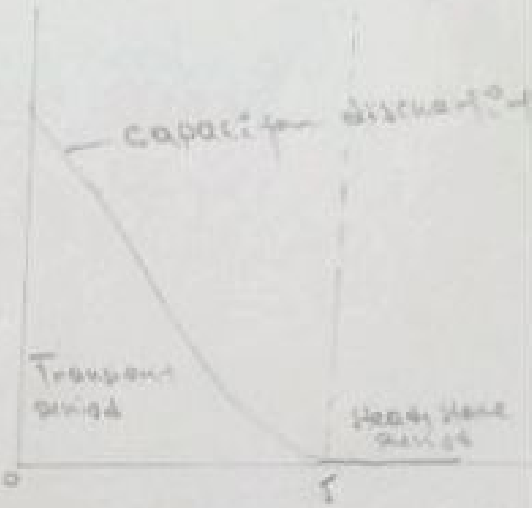
SWITCH



So far source free RC cct let's see capacitor charging and discharging.



Transient period from 0 to 1



Discharging from 0 to 1

③ capacitor voltage cannot change abruptly

① Natural Response of an RL circuit

- A cct that can be simplified to a Thevenin or Norton equivalent connected to either a simple equivalent inductor or capacitor

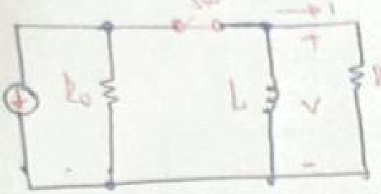
What is natural response?

- It describes the "discharging" of inductor or capacitor via a circuit of no dependent source.

- No external source involved is called "natural response"

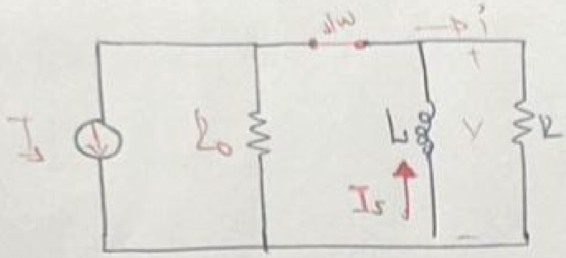
Natural Response of an RL ckts

Let the ckt shown which contains an inductor is shown-



Suppose the switch has been in a closed position for a long time.

- For the time being long time
- All currents and voltages have reached a constant value
- Only constant or DC current can exist in the ckt just prior to the switch's being opened.
- The inductor appears as a short ckt $\boxed{V = L \frac{di}{dt} = 0}$

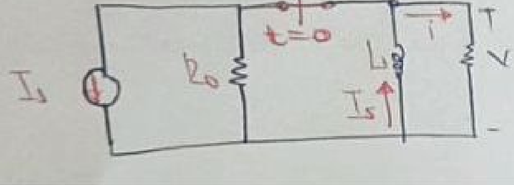


* When the switch is closed, the inductor appears as a short, the voltage across the branch is zero.
* When L appears as a short, no current goes through R_0 & R.
* The current (the source current) I_s all goes through L.

OR

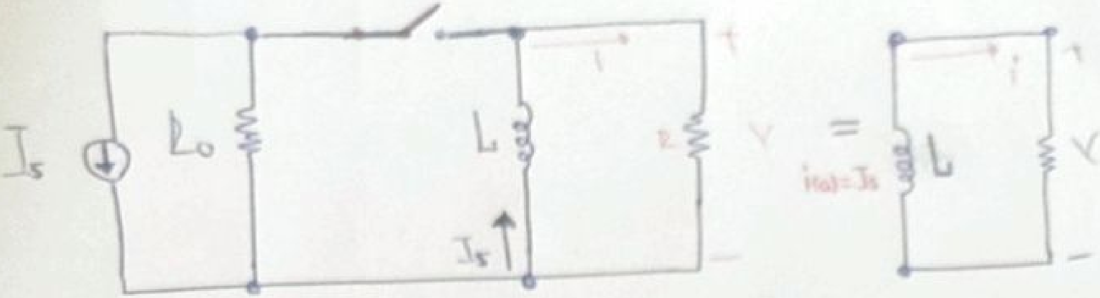
- Because the vtg across the inductive branch is zero
  - no current in either R_0 or R
  - All the source current I_s appears in inductive branch

Finding the natural response requires finding the vtg or current at any branch in the ckt after the switch has been opened.



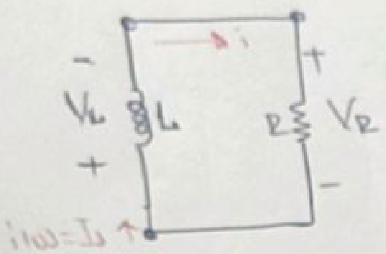
If we let $t=0$ denote the instant when the switch is opened

For $t \geq 0$ the ckt becomes



③ The problem becomes one of finding $v(t)$ & $i(t)$ (ie i & v) for $t \geq 0$

Deriving the Expression for the current



Apply KVL around the loop

$$V_L + V_R = 0$$

$$\boxed{L}\underbrace{\frac{di}{dt}}_{\text{Inductor Law}} + \boxed{R}\underbrace{i}_{\text{Ohm's law}} = 0$$

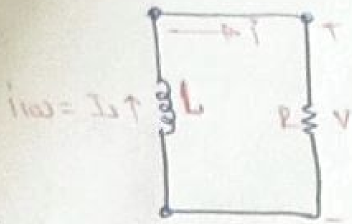
This is a first order DE because it contains terms involving the ordinary derivative of the unknown $\frac{di}{dt}$

- The highest order derivative appearing in the eqn 1 hence 1st order.

- We can still describe the equation further

④ Since the coefficients in equation R & L are constant that is not fns of either the dependent variable i or independent variable t

- Thus the equation can be described as an ODE w constant coefficients



$$L\frac{di}{dt}+Ri=0$$

$$L\frac{di}{dt}=-Ri$$

$$\frac{di}{dt}=-\frac{Ri}{L}$$

$$\frac{di}{\cancel{dt}}\cancel{dt}=-\frac{R}{L}i\,dt$$ [Take the differential with respect to time of both sides of the equation]

$$di=-\frac{R}{L}i\,dt$$

$$\frac{di}{i}=-\frac{R}{L}dt$$

Integrating both side to obtain explicit expression for i as a function of t

$$\int_{i(t_0)}^{i(t)}\frac{dx}{x}=-\frac{R}{L}\int_{t_0}^{t}dy$$ Here $t_0=0$

$$\int_{i(0)}^{i(t)}\frac{dx}{x}=\frac{-R}{L}\int_{0}^{t}dy$$

$$\ln(i(t))-\ln(i(t_0))=-\frac{R}{L}(t-t_0)$$ $t_0=0$

$$\ln(i(t))-\ln(i(0))=-\frac{R}{L}(t)$$

$$\ln\frac{i(t)}{i(0)}=-\frac{R}{L}t$$

Based on the definition of the natural logarithm $\boxed{e^{\ln x}=x}$

- Apply base e to both sides

$$e^{\ln\frac{i(t)}{i(0)}}=e^{-\frac{R}{L}t}$$

$$\frac{i(t)}{i(0)}=e^{-\frac{R}{L}t}$$

$$\boxed{i(t)=i(0)\,e^{-(R/L)t}}$$

- Instantaneous change of current cannot occur in an inductor
- The instant right after the switch is opened, the current in the inductor remains unchanged.

Since the current through an Inductor cannot change abruptly or instantaneously

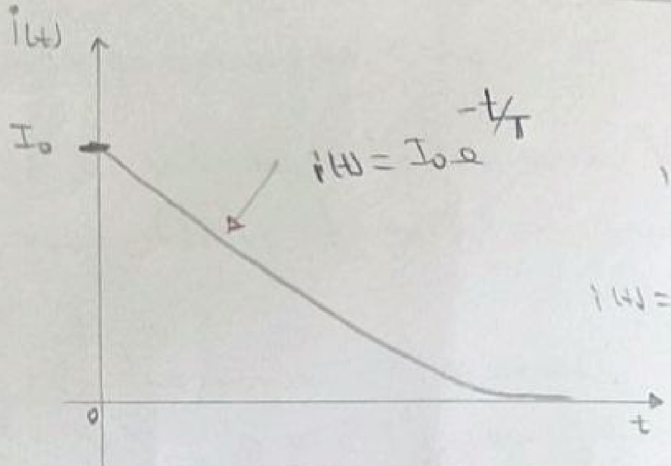
$$\boxed{i(0^-) = i(0^+) = i(0) = I_0}$$

- Use 0^- to denote ~~time~~ prior to switching
- Use 0^+ to denote time immediately after switching

Where I_0 is the initial current on the inductor just before the switch is opened (or in some case closed)

In the ckt above $I_0 = I_s$

Therefore $$\boxed{i(t) = I_0 e^{-(R/L)t} \quad t \geq 0}$$



When $t = 0$, $i(t) = I_0$

$i(t) = I_0 e^{0} = I_0$

When $t = \tau$

$i(t) = I_0 e^{-1} = \frac{I_0}{e}$

Which shows currents starts at the initial value I_0 and decreases exponentially toward zero as t increases.

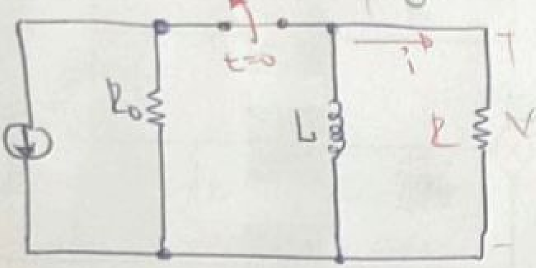
Voltage

- We derive the voltage across the resistor from direct application of Ohm's law

$$V(t) = R\,i(t) \qquad \text{but } i(t) = I_0 e^{-(R/L)t} \quad t \geq 0^+$$

$$\boxed{V(t) = R I_0 e^{-(R/L)t} \qquad t \geq 0^+}$$

Note: The voltage $V(t)$ here is defined for $t \geq 0$ because the voltage across the resistor R is zero for $t<0$ (all the current I_s was going through L & zero current through R)



$$\boxed{\begin{aligned} V(0^-) &= 0 \\ V(0^+) &= I_0 R \end{aligned}}$$

- Voltage is only defined for $t>0$, not at $t=0$ because a step voltage occurs at $t=0$.

- Because of the step voltage the value at $t=0$ is unknown, therefore only $t \geq 0^+$ defines region of validity

Power

- We derive the power dissipated in the resistor

$$p = vi, \quad \boxed{p = Ri^2} \quad \text{or} \quad p = v^2/R$$

$$p = (I_0 e^{-(R/L)t})^2 \times R \quad \text{for } t \geq 0^+$$

Squaring the current yields:

$$\boxed{p = I_0^2 R\, e^{-2(R/L)t} \quad \text{for } t \geq 0^+}$$

$$e^{-(R/L)t} \cdot e^{-(R/L)t} = e^{-(R/L)t + -(R/L)t} = e^{-(R/L)(t+t)} = e^{-2(R/L)t}$$

Note the current $i(t)$ through the resistor is zero for $t<0$

$$p = 0, \quad t<0$$

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Energy:

The energy delivered to the resistor is after the switch is opened because before there was no current passing through the resistor and the voltage across it was zero.

$$W = \int_0^t p\,dx = \int_0^t I_0^2 R\, e^{-2(R/L)x}\,dx = I_0^2 R \int_0^t e^{-2(R/L)x}\,dx$$

$$= I_0^2 R \left[\frac{1}{-2(R/L)} \left(e^{-2(R/L)x} \right) \right]_0^t$$

$$= I_0^2 R \left[\frac{1}{-2(R/L)} \left(e^{-2(R/L)t} - e^{-2(R/L)0} \right) \right]$$

$$= I_0^2 R \left[\frac{1}{-2(R/L)} \left(e^{-2(R/L)t} - 1 \right) \right]$$

$$= \frac{1}{2(R/L)} I_0^2 R \left(1 - e^{-2(R/L)t} \right)$$

$$= \frac{1}{2} L I_0^2 \left(1 - e^{-2(R/L)t} \right) \text{ for } t \geq 0$$

$$\therefore \boxed{W = \frac{1}{2} L I_0^2 \left(1 - e^{-2(R/L)t} \right) \text{ for } t \geq 0}$$

Recap

$$i(t) = I_0 e^{-(R/L)t} \quad \text{for } t \geq 0$$

$$v = I_0 R e^{-(R/L)t} \quad \text{for } t \geq 0^+$$

$$p = I_0^2 R e^{-2(R/L)t} \quad \text{for } t \geq 0^+$$

$$W = \frac{1}{2} L I_0^2 \left(1 - e^{-2(R/L)t} \right) \quad \text{for } t \geq 0$$

$$\boxed{\tau = \frac{L_{eq}}{R_{eq}}}$$

The significant of the time constant

The coefficient of the exponential term namely R/L determine the rate at which the exponential term in the current approaches zero.

Time constant (τ) –, Determine the rate at which the current or Voltage approaches zero.

$$\boxed{\tau = \text{Time constant} = \frac{L}{R} \qquad \tau - \text{Tau}}$$

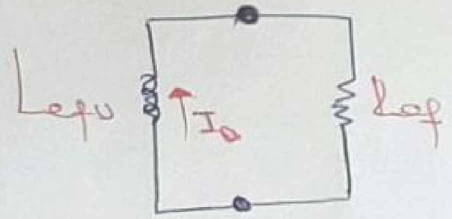
* Time constant for R, L ckt is $\tau = L/R$

Re-write the equation w τ

$$\boxed{\begin{aligned} i(t) &= I_0 e^{(-t/\tau)} \quad \text{for } t \geq 0 \\ V &= I_0 R e^{(-t/\tau)} \quad \text{for } t \geq 0^+ \\ P &= I_0^2 R e^{(-2t/\tau)} \quad \text{for } t \geq 0^+ \\ W &= \frac{1}{2} L I_0^2 (1 - e^{(-2t/\tau)}) \quad \text{for } t \geq 0 \end{aligned}}$$

Time Constant Characteristic

- After 5 time constants, currents & vtgs have reached their final value
- A long time - implies 5 or more time constants have elapsed.
- steady-state response:- the response that exists a long time after switching has taken place.
- The existence of current in the RL ckt is a momentary event and is called a transient response.



At 5 time constants current is less than 1% to of it's initial value (I_0)

t	$e^{-(t/\tau)}$
T	3.6188×10^{-1}
$2T$	1.3534×10^{-1}
$3T$	4.9781×10^{-2}
$4T$	1.8316×10^{-2}
$5T$	6.7379×10^{-3}
$6T$	2.4788×10^{-3}
⋮	
$10T$	4.540×10^{-5}

$$I(t) = I_0 e^{-(t/\tau)}$$

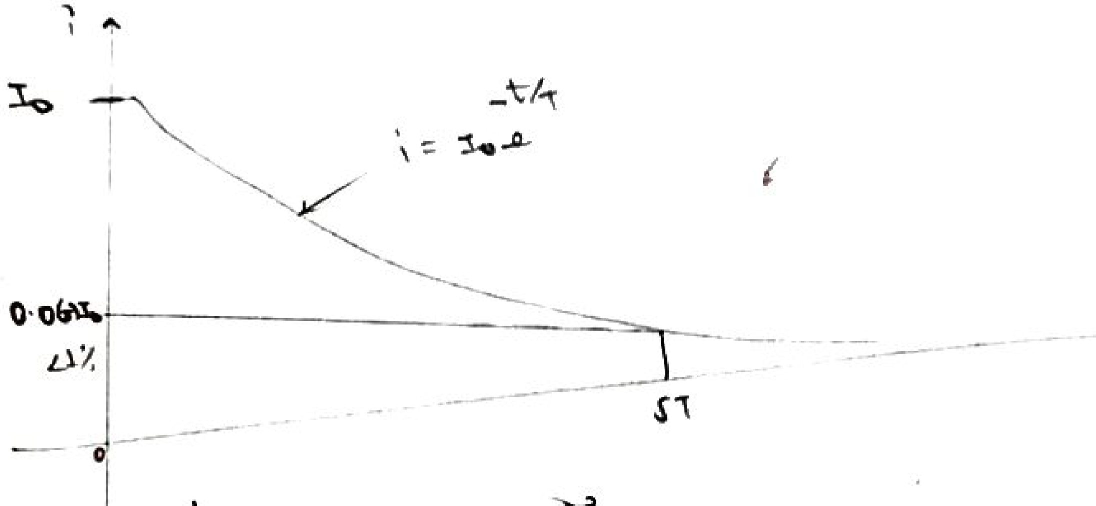
when $t = T$

$$i(T) = I_0 e^{-(T/\tau)}$$

$$i(T) = I_0 e^{-1}$$

when $t = 2T$

$$i(2T) = I_0 e^{-(2T/\tau)} = I_0 e^{-2}$$

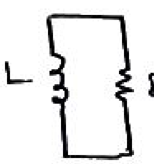


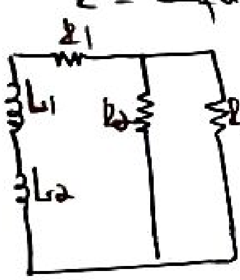
Determining the time constant T

(1) If the RL ckt can be put as

L - equivalent Inductor

R - equivalent Resistor





$\Rightarrow$ $L_{eq} = L_1 + L_2$

$R_{eq} = R_1 + (R_2 \| R_3)$

$$\boxed{\tau = \frac{L_{eq}}{R_{eq}}}$$

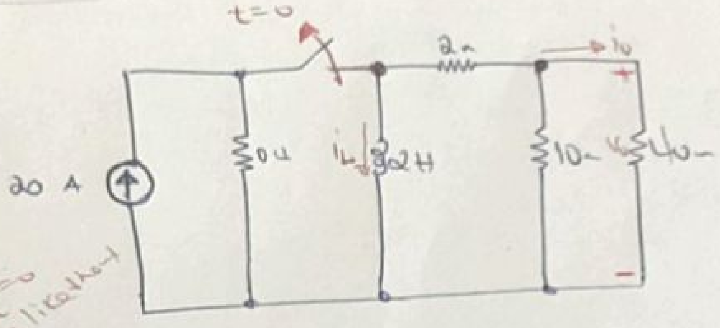
Steps to calculate the natural response of an RL ckt

#1. Find the initial current, I_0, through the inductor

#2. Find the time constant of the ckt, $\tau = L/R$

#3. Use $i(t) = I_0 e^{-t/\tau}$ to calculate $i(t)$

Example: The switch has been closed for a long time before it is opened at $t=0$. Find $i_L(t)$, $i_0(t)$, $v_0(t)$, & the percentage of the total energy stored in the 2H inductor that is dissipated in the 10Ω resistor



Sol.

steady-state condition for $t<0$
$V_L = L\frac{di}{dt} = 0$
inductor as if short ckt.

Since the switch has been closed a long period of time, the inductor appears as a short ckt, concluding that the initial current through the inductor is $\underline{20A}$

- The equivalent resistance across the terminals of the inductor will be . $R_{eq} = 2 + (10 \| 40) = 2 + \frac{10\times 40}{10+40} = 2+8 = \underline{10}$ ohms

. Time constant $= \tau = \frac{L}{R_{eq}} = \frac{2}{10} = \underline{0.2\,s}$

$$i_L(t) = I_0 e^{-t/\tau} = 20 e^{-t/0.2} = 20 e^{-5t}\ A$$

$$i_L(t) = \underline{20 e^{-5t}}\ A$$

Now calculate i_0 through 40Ω resistor
using a current divider

$$i_0(t) = -i_L\left(\frac{10}{10+40}\right) = -20e^{-5t}\left(\frac{1}{5}\right) = \underline{-4e^{-5t}}\ A$$

Using Ohm's law

$V_o = i_o R$ but $i_o = -4e^{-5t}$ A

$V_o(t) = (-4e^{-5t})(40)$

$V_o(t) = -160e^{-5t}$ V

⊛ Find the percentage of the total energy stored in the 2H inductor that is dissipated in the 10Ω resistor

$V_o(t) = -160e^{-5t}$ V

$$P_{10\Omega}(t) = \frac{V_o^2}{10} = \frac{(-160e^{-5t})^2}{10} = 2560e^{-10t}\ \text{W}$$

Total energy dissipated in the 10Ω resistor

$$W_{10\Omega}(t) = \int_0^\infty p\,dt = \int_0^\infty 2560e^{-10t}\,dt = \frac{2560}{-10}e^{-10t}\Big|_0^\infty$$

$$= \frac{2560}{-10}\left(e^{-10(\infty)} - e^{-10(0)}\right)$$

$$= \frac{2560}{-10}(0-1)$$

$$= 256\text{J}$$

⊛ Initial energy stored in the 2H inductor

$$W(0) = \frac{1}{2}Li^2(0) = \frac{1}{2}(2)(20)^2 = 400\text{J}$$

⊛ The percentage of energy dissipated in the 10Ω resistor is

$$\frac{256}{400} \times (100) = 64\%$$